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Embeddings of homology equivalent manifolds with boundary

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Abstract

We prove a theorem on equivariant maps implying the following two corollaries:

(1) Let N and M be compact orientable n-manifolds with boundaries such that $M \subset N$, the inclusion $M \to N$ induces an isomorphism in integral cohomology, both M and N have $(n-d-1)$-dimensional spines and $m \geqslant \max\{n+3, \frac{3n+2-d}{2}\}$. Then the restriction-induced map $\mathrm{Emb}^m(N) \to \mathrm{Emb}^m(M)$ is bijective. Here $\mathrm{Emb}^m(X)$ is the set of embeddings $X \to \mathbb{R}^m$ up to isotopy (in the PL or smooth category).

(2) For a 3-manifold N with boundary whose integral homology groups are trivial and such that $N \ncong D^3$ (or for its special 2-spine N) there exists an equivariant map $\widetilde{N} \to S^2$, although N does not embed into $\mathbb{R}^3$.

The second corollary completes the answer to the following question: for which pairs (m,n) for each n-polyhedron N the existence of an equivariant map $\widetilde{N} \to S^{m-1}$ implies embeddability of N into $\mathbb{R}^m$? An answer was known for each pair (m,n) except $(3,3)$ and $(3,2)$.

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This paper is on the classical problem of classification of embeddings into Euclidean spaces, and as a main tool we use the Haefliger–Wu invariant. We begin with the formulation of our main homotopy result. Let $\widetilde{N} = \{(x, y) \in N \times N \mid x \neq y\}$ be the *deleted product* of N. Let $\mathbb{Z}_2$ act on $\widetilde{N}$ and on S^{m-1} by exchanging factors and antipodes, respectively. Denote by $\pi_{\text{eq}}^{m-1}(\widetilde{N})$ be the set of equivariant maps $\widetilde{N} \to S^{m-1}$ up to equivariant homotopy. The set $\pi_{\text{eq}}^{m-1}(\widetilde{N})$ can be effectively calculated [4, beginning of §2], [6], [7, 1.7.1], [20], [1, 7.1], [19, §6]. Note that $\pi_{\text{eq}}^{m-1}(\widetilde{N}) = \emptyset$ for $m < n$ because $\widetilde{N} \supset \widetilde{D^n} \simeq_{\text{eq}} S^{n-1}$.

Theorem. *Let N and M be compact orientable connected n-manifolds with non-empty boundaries such that $M \subset N$ and the inclusion $M \to N$ induces an isomorphism in cohomology with $\mathbb{Z}$-coefficients. Then the restriction-induced map $\pi_{\text{eq}}^{m-1}(\widetilde{N}) \to \pi_{\text{eq}}^{m-1}(\widetilde{M})$ is bijective.*

More precisely, if the inclusion-induced homomorphism $H^i(N) \to H^i(M)$ is an isomorphism only for $i \geqslant l > 0$, then the above restriction-induced map is bijective for $m \geqslant n + l$ and surjective for $m = n + l - 1$.

This algebraic result is interesting because of the following geometric corollaries. Denote CAT = DIFF or PL. For a CAT manifold N let $\text{Emb}_{\text{CAT}}^m(N)$ be the set of CAT embeddings $N \to \mathbb{R}^m$ up to CAT isotopy. A folklore general conjecture, supported by many known results (for a survey see, e.g., [12,20]) is that $\text{Emb}_{\text{CAT}}^m(N)$ is not changed under homology equivalence of N (i.e., under a map $f : M \to N$ between manifolds inducing an isomorphism in (co)homology), in the PL case for $m \geqslant n + 3$ and in the DIFF case for $m \geqslant \frac{3n}{2} + 2$.

Corollary. *Let N and M be compact orientable n-manifolds with non-empty boundaries such that $M \subset N$, the inclusion $M \to N$ induces an isomorphism in cohomology with $\mathbb{Z}$-coefficients, both M and N has $(n - d - 1)$-dimensional spine and $m \geqslant \max\{n + 3, \frac{3n+2-d}{2}\}$. Then the restriction-induced map $\text{Emb}_{\text{CAT}}^m(N) \to \text{Emb}_{\text{CAT}}^m(M)$ is bijective.*

More precisely, if the inclusion-induced homomorphism $H^i(N) \to H^i(M)$ is an isomorphism only for $i \geqslant l > 0$, then the above restriction-induced map is bijective for $m \geqslant n + l$ and surjective for $m = n + l - 1$.

In the DIFF category the restriction $m \geqslant \frac{3n+2-d}{2}$ can be relaxed to $m \geqslant \frac{3n+1-d}{2}$.

By the Corollary, any homology ball unknots in codimension at least 3 (cf. [14]).

Recall that a subpolyhedron K of a manifold N is called a *spine* of N if N is a regular neighborhood of K in N (or, equivalently, if N collapses to K) [13]. We remark that for a compact connected n-manifold N with boundary, the property of having an $(n - d - 1)$-dimensional spine is close to d-connectedness of $(N, \partial N)$. Indeed, for a compact connected n-manifold N with boundary and an $(n - d - 1)$-dimensional spine, the pair $(N, \partial N)$ is homologically d-connected. On the other hand, every compact connected

n-manifold N with boundary for which $(N, \partial N)$ is d-connected, $\pi_1(\partial N) = 0$, $d + 3 \leqslant n$ and $(n, d) \notin \{(5, 2), (4, 1)\}$, has an $(n - d - 1)$-dimensional spine [22, Theorem 5.5], [8, Lemma 5.1 and Remark 5.2]. Recall that a closed manifold N (or a pair $(N, \partial N)$) is called *homologically d-connected*, if N is connected and $H_i(N) = 0$ for each $i = 1, \dots, d$ (or $H_i(N, \partial N) = 0$ for each $i = 0, \dots, k$).

The Corollary follows from the Theorem and the bijectivity of α-invariant [7, 12, §4], [19, Theorems $1.1\alpha\partial$ and $1.3\alpha\partial$], which is defined as follows. For an embedding $f : N \to \mathbb{R}^m$ define a map

$$\tilde f : \widetilde N \to S^{m-1} \quad \text{by} \quad \tilde f(x, y) = \frac{fx - fy}{|fx - fy|}.$$

The equivariant homotopy class $\alpha(f)$ of the above-defined $\tilde f$ in $\pi^{m-1}_{\mathrm{eq}}(\widetilde N)$ is clearly an isotopy invariant. Thus is defined the *Haefliger–Wu* (*deleted product*) *invariant*

$$\alpha = \alpha^m_{\mathrm{CAT}}(N) : \mathrm{Emb}^m_{\mathrm{CAT}}(N) \to \pi^{m-1}_{\mathrm{eq}}(\widetilde N).$$

We remark that the assumption that (N, M) is a codimension 0 pair is essential in the Theorem and the Corollary. Indeed, take $N = D^p \times S^q$ and $M = S^q$. For $m \geqslant 3q/2 + 2$ we have $\#\mathrm{Emb}^m(S^q) = 1$ while $\mathrm{Emb}^m(D^p \times S^q) = \pi_q(V_{m-q,p})$ can contain more than one element (specific examples are particularly easy to find for $p = 1$, when $V_{m-q,p} \simeq S^{m-q-1}$).

The assumption that N has boundary is not essential in the Theorem and the Corollary. But these results are trivial for closed N: if N is closed and $M \neq N$, then the assumptions are never fulfilled because $H^n(N) \neq H^n(M)$. Note that the conclusion of the Theorem for closed manifolds is not always fulfilled, for there are closed manifolds non-embeddable in the same dimension as the corresponding punctured manifolds.

Now let us present motivation for the second corollary of the Theorem. From the construction of the map $\tilde f$ above it follows that

$$\text{if } N \text{ embeds into } \mathbb{R}^m, \text{ then there exists an equivariant map } \widetilde N \to S^{m-1}. \tag{$*$}$$

The existence of an equivariant map $\widetilde N \to S^{m-1}$ can be checked for many cases [4, beginning of §2], [6], [7, 1.7.1], [20], [1, 7.1]. Thus if a converse to $(*)$ is true, the embedding problem is reduced to manageable (although not trivial) algebraic problems. So in 1960s there appeared a problem to find conditions under which the converse to $(*)$ is true. The converse for $(*)$ was known to be

true for an n-polyhedron N and $2m \geqslant 3n + 3$ *or* $m = 2n = 2$
[7,25,23], see also [17,18], [12, §4], [19].

false for each pair (m, n) *such that* $\max\{4, n\} \leqslant m \leqslant \frac{3n}{2} + 1$ *and some n-polyhedron N* [10,9,15,5,21].

In the only remaining cases $m = 3$ and $n \in \{2, 3\}$ it was unknown if the converse to $(*)$ is true. The counterexamples to the converse of $(*)$ for $m = n \geqslant 4$ and $m = n + 1 \geqslant 4$ [10,9] cannot be directly extended to $m = 3$ because they used m-dimensional (Mazur) contractible manifolds distinct from the m-ball, which apparently do not exist for $m = 3$.

Recall that a *homology n-ball* is an n-manifold with boundary whose homology groups are the same as those of the n-ball.

Proposition. *The converse to* $(*)$ *is false in the cases* $m = 3$ *and* $n \in \{2, 3\}$: *if* N *is either a non-trivial homology ball or its special spine, then* N *does not embed into* $\mathbb{R}^3$ *but there exists an equivariant map* $\widetilde{N} \to S^2$.

Proof. The non-embeddability follows because if a special spine of a homology ball N embeds into $\mathbb{R}^3$, then the regular neighborhood in $\mathbb{R}^3$ of this spine is homeomorphic to N [3], which contradicts to the non-triviality of N.

It suffices to prove the existence of an equivariant map $\widetilde{N} \to S^2$ for a homology 3-ball N. For this case this existence follows from the Theorem because an inclusion of the standard ball into N induces an isomorphism in cohomology.

We also present an alternative proof of the existence of an equivariant map $\widetilde{N} \to S^2$ for a homology 3-ball N, which proof is shorter than that of Theorem. Analogously to [1, end of §7.1] (or by Steps 2 and 3 below) it suffices to prove that $H^i(\widetilde{N}; \mathbb{Z}) = 0$ for each $i \geqslant 3$. We prove this for $i = 3$ (the proof for each $i \geqslant 4$ is analogous). We omit $\mathbb{Z}$-coefficients from the notation. Let V be a closed regular neighborhood in $N \times N$ of the diagonal. Then

$$\begin{aligned} H^3(\widetilde{N}) &\cong H^3(N \times N - \mathring{V}) \cong H_3\big(N \times N - \mathring{V}, \partial(N \times N - \mathring{V})\big) \\ &\cong H_3\big(N \times N, V \cup \partial(N \times N)\big) \cong H_2\big(V \cup \partial(N \times N)\big) = 0. \end{aligned}$$

Here the first isomorphism follows because $N \times N - \mathring{V}$ is a deformation retract of $\widetilde{N}$, the second one by Lefschetz duality (recall that $\widetilde{N}$ is orientable if N a homology ball), the third one by excision, and the fourth one by exact sequence of pair. Using the Mayer–Vietoris sequence for

$$\partial(N \times N) = N \times \partial N \bigcup_{\partial N \times \partial N} \partial N \times N$$

and noting that $\partial N \cong S^2$, we prove that $H_2(\partial(N \times N)) = 0$. Using the Mayer–Vietoris sequence for $V \cup \partial(N \times N)$ and noting that $V \simeq N$ and $V \cap \partial(N \times N)$ is a regular neighborhood in $N \times N$ of the diagonal of ∂N, i.e. is homotopy equivalent to $\partial N \cong S^2$, we prove the last isomorphism. □

Another proof of the Proposition can be obtained by using the fact that for the homology 3-ball N, which is a punctured boundary of the Mazur 4-manifold, there exists an equivariant map $\Sigma \widetilde{N} \to S^3$ [11]. Indeed, one can show that the obstruction to equivariant desuspension of this map on $\widetilde{P}$ (where P is the special spine of N) lies in $H^4(\widetilde{P}, \mathbb{Z})$, which group is trivial because P is acyclic [24]. We do not present the details because we already have a complete proof of the Proposition.

Proof of the Theorem (without the 'more precisely' part). Denote $N^* = \widetilde{N}/(x, y) \sim (y, x)$. Consider the bundle

$$g : \frac{\widetilde{N} \times S^{m-1}}{(x, y, s) \sim (y, x, -s)} \xrightarrow{S^{m-1}} N^*, \quad \text{given by } g\big[(x, y, s)\big] = \big[(x, y)\big].$$

Equivariant maps $\widetilde{N} \to S^{m-1}$ up to equivariant homotopy are in 1–1 correspondence with cross-sections of g up to equivalence [4, beginning of §2], [6], [7, 1.7.1], [20], [1, 7.1]. So we need to prove that any section on M^* can be extended to a section on N^*, and such an extension is unique up to equivalence. This follows because below (in Steps 1, 2 and 3) we prove that

$$H^{i+1}\big(N^*, M^*; \pi_i(S^{m-1})_{tw}\big) \cong 0 \quad \text{and}$$
$$H^{i}\big(N^*, M^*; \pi_i(S^{m-1})_{tw}\big) \cong 0 \quad \text{for each } i.$$

Here the coefficients are twisted according to the double cover $\widetilde{N} \to N^*$ and the involution on $\pi_i(S^{m-1})$ induced by the antipodal involution on S^{m-1} (so the twisting is trivial for m even).

Step 1. *Proof that* $H^i(\widetilde{N}, \widetilde{M}; G) \cong 0$ *for each* i *and any non-twisted* G. Let N_0 and M_0 the interiors of N and M, respectively. The collaring theorem for the boundary of a manifold says that there is a neighborhood of ∂M in M which is homeomorphic to the product $\partial M \times [0,1)$ so that $\partial M \times \{0\}$ is mapped homeomorphically to the boundary. Therefore there is an embedding of $\phi : N \to N_0$ which is a homotopy inverse of the inclusion $i : N_0 \to N$. In a similar fashion the map $\phi \times \phi : \widetilde{N} \to \widetilde{N}_0$ is the homotopy inverse of the inclusion $\widetilde{N}_0 \to \widetilde{N}$. Analogous observations hold for N replaced by M. So it suffices to prove Step 1 for N and M replaced by N_0 and M_0.

Let $x_0 \in M_0 \subset N_0$ be a base point for M_0 and N_0. Consider the following mapping of bundles (which are given by projections onto the first factor):

$$\begin{array}{ccccc}
M_0 - x_0 & \longrightarrow & \widetilde{M}_0 & \longrightarrow & M_0 \\
\downarrow \subset & & \downarrow \subset & & \downarrow \subset \\
N_0 - x_0 & \longrightarrow & \widetilde{N}_0 & \longrightarrow & N_0
\end{array}$$

Consider the spectral sequences associated to the bundles above. Observe that $M_0 - x_0 \simeq M_0 \vee S^{n-1}$. Hence the action of $\pi_1(M_0)$ in the (co-)homology $H^i(M_0 - x_0)$ is trivial for all i (the same holds for the second bundle, where M is replaced by N). This follows for $i = n$ because $H^n(M_0 - x_0) = 0$ and for $i < n-1$ because the projection onto the first factor is the restriction of the trivial bundle $M_0 \times M_0 \to M_0$, while $H^i(M_0 - x_0) \cong H^i(M_0)$. For $i = n-1$ we have $H^{n-1}(M_0 - x_0) \cong H^{n-1}(M_0) \oplus \mathbb{Z}$. The action of an element $\alpha \in \pi_1(M_0)$ is given by the identity on the first summand and multiplication by the sign of the loop on $\mathbb{Z}$. Since the manifolds are orientable, the action is identical.

Consider the restriction-induced homomorphism

$$j : H^p\big(N_0; H^q(N_0 - x_0; G)\big) \to H^p\big(M_0; H^q(M_0 - x_0; G)\big)$$

of the E_2-terms of the Leray–Serre cohomology spectral sequences of the above bundles. By excision the inclusion of the pairs $(M_0, M_0 - x_0) \to (N_0, N_0 - x_0)$ induces an isomorphism in cohomology. Applying 5-lemma for the inclusion-induced mapping of long exact sequences of these pairs we obtain that the inclusion $M_0 - x_0 \to N_0 - x_0$ induces an isomorphism in cohomology. Hence using the triviality of the action and the Universal Coefficients Theorem we obtain that the restriction-induced homomorphism j is an isomorphism for all p and q. By the Zeeman Comparison Theorem of spectral sequences [26], the restriction $H^i(\widetilde{N}_0; G) \to H^i(\widetilde{M}_0; G)$ is an isomorphism. This implies Step 1.

A statement on cohomology of compact manifolds should have a proof involving only cohomology of compact manifolds (recall that we may assume that $\widetilde{N} = \widetilde{N}_\varepsilon$ is compact). The above proof has such an interpretation in terms of only compact spaces. Step 1 can also be proved analogously to proof of the Proposition above. We do not present details in order to spare space.

Step 2. *Proof that* $H^i(N^*, M^*; G) \cong 0$ *for each* i *and any non-twisted coefficient system* G. Consider the double coverings $\widetilde{M} \to M^*$ and $\widetilde{N} \to N^*$. The restriction induces a map of their Cartan–Leray spectral sequences, see [2, Chapter XVI, Theorem 8.4 on p. 354]. For the E_2-terms we have a map

$$E_2^{p,q}(N^*) = H^p\big(\mathbb{Z}_2, H^q(\widetilde{N})\big) \to E_2^{p,q}(M^*) = H^p\big(\mathbb{Z}_2, H^q(\widetilde{M})\big),$$

which is an isomorphism by Step 1 and the exact sequence of pair $(\widetilde{N}, \widetilde{M})$. Hence the restriction induces an isomorphism $H^i(N^*; G) \to H^i(M^*; G)$, and the result follows.

Another proof of Step 2. The $\mathbb{Z}_2$ principal fibration $\widetilde{M} \to M^*$ is the induced by the inclusion $i : M^* \to N^*$ from the $\mathbb{Z}_2$ principal fibration $\widetilde{N} \to N^*$. The classifying map of the principal fibration $\widetilde{N} \to N^*$ is a map $f : N^* \to \mathbb{R}P^\infty$, and the classifying map for $\widetilde{M} \to M^*$ is the composition $f \circ i$. Therefore we obtain a commutative diagram of bundles

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d, "\subset"] & E_1 \arrow[r] \arrow[d, "\subset"] & \mathbb{R}P^\infty \arrow[d, "="] \\ F_2 \arrow[r] & E_2 \arrow[r] & \mathbb{R}P^\infty \end{tikzcd}$$

where F_1, E_1, F_2 and E_2 have the same homotopy type as $\widetilde{M}$, M^*, $\widetilde{N}$ and N^*, respectively. Now Step 2 follows because for the E_2-term of cohomology spectral sequence of this pair of bundles we have $E_2^{p,q} = H^p(\mathbb{R}P^\infty; H^q(E_2, E_1; G)) \cong 0$ for each p, q and any non-twisted G.

Step 3. *Proof that* $H^i(N^*, M^*; G_{tw}) \cong 0$ *for each* i *and a local coefficient system* G_{tw} *associated to the double cover* $\widetilde{N} \to N^*$ *and certain involution* $\varphi : G \to G$ *of a module* G*, which is a finitely-generated Abelian group.* It suffices to consider the case when $G = \mathbb{Z}$. The local system induced by G_{tw} on $\pi_1(\widetilde{N})$ is non-twisted. By naturality, G_{tw} induces a local system in M^*. Clearly, the local system over M^* also induces a trivial system on $\widetilde{M}$, and the local system on M^* is trivial if and only G_{tw} is trivial on N^*. Observe that there is only one non-trivial local system $\mathbb{Z}_{tw}$ on N^* which induces the trivial local system on $\widetilde{N}$, and the same for M^*. Now consider the map between the Smith–Richardson–Thom–Gysin sequences associated with the double covers $\widetilde{M} \to M^*$ and $\widetilde{N} \to N^*$ (see the Smith–Richardson–Thom–Gysin Sequence Theorem below, cf. [1, §7]).

$$\begin{tikzcd} H^j(N^*, \mathbb{Z}) \arrow[r] \arrow[d] & H^j(\widetilde{N}, \mathbb{Z}) \arrow[r] \arrow[d] & H^j(N^*, \mathbb{Z}_{tw}) \arrow[r] \arrow[d] & H^{j+1}(N^*, \mathbb{Z}) \arrow[r] \arrow[d] & H^{j+1}(\widetilde{N}, \mathbb{Z}) \arrow[d] \\ H^j(M^*, \mathbb{Z}) \arrow[r] & H^j(\widetilde{M}, \mathbb{Z}) \arrow[r] & H^j(M^*, \mathbb{Z}_{tw}) \arrow[r] & H^{j+1}(M^*, \mathbb{Z}) \arrow[r] & H^{j+1}(\widetilde{M}, \mathbb{Z}) \end{tikzcd}$$

Suppose by induction that $H^{j+1}(N^*, \mathbb{Z}_{tw}) \to H^{j+1}(M^*, \mathbb{Z}_{tw})$ is an isomorphism. Certainly this is true for $j + 1$ big enough so we can start the induction process. Then the inductive step follows by Steps 1, 2 and the five lemma.

The proof of the 'more precisely' part of the Theorem. Now assume that the inclusion $M \to N$ induces isomorphism in cohomology only down to dimension l. It suffices to prove, Steps 1, 2 and 3 for $i \geqslant n+l$ instead of arbitrary i. We show how to do this for Step 1, and the required changes in the proofs of Steps 2 and 3 are obvious. The condition $H^i(\widetilde{N}, \widetilde{M}; G) \cong 0$ for $i \geqslant n+l$ is equivalent to $H^i(\widetilde{N}_0, \widetilde{M}_0; G) \cong 0$ for $i \geqslant n+l$. Consider the spectral sequences associated to the bundles given in Step 1. Recall that

$$H^n(N_0 - x_0; G) = H^n(N_0; G) = H^n(M_0 - x_0; G) = H^n(M_0; G) = 0$$

for any coefficients G. As in Step 1 the action of $\pi_1(M_0)$ in the (co-)homology $H^i(M_0 - x_0)$ is trivial for all i. Analogous statement hods for the second fibration. By excision the inclusion of the pairs $(M_0, M_0 - x_0) \to (N_0, N_0 - x_0)$ induces an isomorphism in cohomology. Applying the 5-lemma for the inclusion-induced mapping of long exact sequences of these pairs we obtain that the inclusion $M_0 - x_0 \to N_0 - x_0$ induces an isomorphism in cohomology for $i \geqslant l$. Using all the facts above and the Universal Coefficient Theorem we see that the restriction induces an isomorphism

$$j : H^p\big(N_0; H^q(N_0 - x_0; G)\big) \to H^p\big(M_0; H^q(M_0 - x_0; G)\big)$$

of the E_2-terms of the Leray–Serre cohomology spectral sequences of the above bundles for $p+q \geqslant n+l-1$. So the restriction induces an isomorphism of $E_2^{p,q}$ for $p+q \geqslant n+l$ and an epimorphism for $p+q = n+l-1$. Now using standard argument of homological algebra as in the Zeeman Comparison Theorem of spectral sequences [26] we obtain that the homomorphism between the terms $E_r^{p,q}$ is an isomorphism for $p+q \geqslant n+l$ and an epimorphism for $p+q = n+l-1$. Since $E_{n-l} = E_{n-l+1} = \cdots = E_\infty$, it follows that the restriction induces on E_∞ terms an isomorphism for $p+q \geqslant n+l$ and an epimorphism for $p+q = n+l-1$. Hence the restriction $H^i(\widetilde{N}; G) \to H^i(\widetilde{M}; G)$ is an isomorphism for $i \geqslant n+l$ and an epimorphism for $p+q = n+l-1$. Therefore by the long exact sequence of pair it follows that $H^i(\widetilde{N}, \widetilde{M}; G) \cong 0$ for $i \geqslant n+l$. $\square$

For the reader's convenience we present below the following slight extension of known Smith–Richardson–Thom–Gysin sequence.

Smith–Richardson–Thom–Gysin Sequence Theorem. *Let X be a connected space, $X' \to X$ a double covering and G a module with an involution ϕ. Consider the local coefficient system G_ϕ on X associated to the double covering and the involution ϕ, the system $G_{-\phi}$ on X associated to double covering p and involution $-\phi$, and the trivial local system G on X'. Then there is a long exact sequence*

$$\cdots \to H^{p-1}(X, G_\phi) \to H^p(X, G_{-\phi}) \to H^p(X', G) \\ \to H^p(X, G_\phi) \to H^{p+1}(X, G_{-\phi}) \to \cdots.$$

If 2 is invertible in G (in particular, if either $G = \mathbb{Q}$ or $G = \mathbb{Z}_p$ for p an odd prime), then we have splittable short exact sequence

$$0 \to H^p(X, G_{-\phi}) \to H^p(X', G) \to H^p(X, G_\phi) \to 0 \quad \textit{so that}$$
$$H^p(X', G) \cong H^p(X, G_{-\phi}) \oplus H^p(X, G_\phi).$$

If $G=\mathbb{Z}$ and $\phi=\mathrm{id}$, then we get long exact sequence

$$\cdots \to H^{p-1}(X,\mathbb{Z}) \to H^p(X,\mathbb{Z}_{tw}) \to H^p(X',\mathbb{Z}) \\ \to H^p(X,\mathbb{Z}) \to H^{p+1}(X,\mathbb{Z}_{tw}) \to \cdots.$$

If $G=\mathbb{Z}$ and $\phi=-\mathrm{id}$, then we get long exact sequence

$$\cdots \to H^{p-1}(X,\mathbb{Z}_{tw}) \to H^p(X,\mathbb{Z}) \to H^p(X',\mathbb{Z}) \\ \to H^p(X,\mathbb{Z}_{tw}) \to H^{p+1}(X,\mathbb{Z}) \to \cdots.$$

Proof. The given local systems $\pi_1(X)\to \mathrm{Aut}(G)$ on X factor through $\mathbb{Z}_2$, so for simplicity we will regard G as a $\mathbb{Z}[\mathbb{Z}_2]$ module. Consider the fibration $F\to X'\to X$ which is the double covering, where F is a two-points set. For the spectral sequence with local coefficients [16] we have $E_2^{p,q}=H^p(X,H^q(F,G))$. But these groups are trivial for each $q>0$. Then the spectral sequence contains at most one non-vanishing line. Hence

$$H^p(X',G_\phi)\cong E_\infty^{p,0}\cong E_2^{p,0}\cong H^p\big(X,(G\oplus G)_{tw}\big),$$

where the local coefficient system $(G\oplus G)_{tw}$ is associated to the double covering and the involution $\phi\oplus\phi$. Namely, the local coefficient system is defined by the automorphism τ of $H^0(F,G)\cong G\oplus G$ induced by the map $F\to F$ permuting the coordinates and by the homomorphism $\phi\oplus\phi$. This homomorphism τ is given by $\tau(a,b)=(\phi(b),\phi(a))$.

Now let $H=\{(m,-m)\in G\oplus G \mid m\in G\}$. The subgroup H is invariant under the action of $\mathbb{Z}[\mathbb{Z}_2]$ on $G\oplus G$, so H is a sub-module. The involution on this submodule is given by $\tau(m,-m)=(\phi(-m),\phi(m))=(-\phi(m),\phi(m))$. Hence H as a module is isomorphic to G with the involution $-\phi$, i.e. to $G_{-\phi}$. The quotient $(G\oplus G)/G_\phi$ is also a $\mathbb{Z}[\mathbb{Z}_2]$ module and it is easy to see that it is isomorphic to G_ϕ. Now the first part of the Smith–Richardson–Thom–Gysin Sequence Theorem follows from the long exact sequence in cohomology associated with the short exact sequence of coefficients of $\mathbb{Z}[\mathbb{Z}_2]$-modules.

The further part where 2 is invertible follows from the fact that the short exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$ modules

$$0\to H\to G\oplus G\to (G\oplus G)/H\to 0$$

splits. Namely, the $\mathbb{Z}[\mathbb{Z}_2]$ homomorphism $s:(G\oplus G)/H\cong G\to G\oplus G$ defined by $s(m)=1/2(m,m)$ is a splitting. The part where $G=\mathbb{Z}$ is clear. $\square$

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References

[1] M. Adachi, Embeddings and Immersions, Transl. Math. Monographs, vol. 124, Amer. Math. Soc., Providence, RI, 1993.

[2] H. Cartan, S. Eilenberg, Homological Algebra, Oxford University Press, Princeton University Press, Oxford–Princeton, 1956.

[3] B.G. Casler, An embedding theorem for connected 3-manifolds with boundary, Proc. Amer. Math. Soc. 16 (1965) 556–559.

[4] P.E. Conner, E.E. Floyd, Fixed points free involutions and equivariant maps, Bull. Amer. Math. Soc. 66 (1960) 416–441.

[5] M.H. Freedman, V.S. Krushkal, P. Teichner, Van Kampen's embedding obstruction is incomplete for 2-complexes in $\mathbb{R}^4$, Math. Res. Lett. 1 (1994) 167–176.

[6] A. Haefliger, Plongements de variétés dans le domain stable, Seminare Bourbaki 245 (1962).

[7] A. Haefliger, Plongements différentiables dans le domain stable, Comment. Math. Helv. 37 (1962–63) 155–176.

[8] K. Horvatič, Embedding manifolds with low-dimensional spine, Glas. Mat. 4(24) (1) (1969) 101–116.

[9] L.S. Husch, ε-maps and embeddings, in: General Topological Relations to Modern Analysis and Algebra, VI, Heldermann, Berlin, 1988, pp. 273–280.

[10] S. Mardešić, J. Segal, ε-mappings and generalized manifolds, Michigan Math. J. 14 (1967) 171–182.

[11] J. Malešič, D. Repovš, A. Skopenkov, Embeddings into $\mathbb{R}^m$ and the deleted product obstruction, Bol. Soc. Mat. Mexicana 9 (2003) 165–170.

[12] D. Repovš, A. Skopenkov, New results on embeddings of polyhedra and manifolds into Euclidean spaces, Uspekhi Mat. Nauk 54 (6) (1999) 61–109, in Russian; English transl.: Russian Math. Surveys 54 (6) (1999) 1149–1196.

[13] C.P. Rourke, B.J. Sanderson, Introduction to Piecewise-Linear Topology, Ergeb. Math., vol. 69, Springer-Verlag, Berlin, 1972.

[14] M. Scharlemann, Isotopy and cobordism of homology spheres in spheres, J. London Math. Soc., Ser. 2 16 (3) (1977) 559–567.

[15] J. Segal, S. Spież, Quasi-embeddings and embedding of polyhedra in $\mathbb{R}^m$, Topology Appl. 45 (1992) 275–282.

[16] J. Siegel, Higher order cohomology operations in local coefficient theory, Amer. J. Math. 89 (1967) 909–931.

[17] A.B. Skopenkov, On the deleted product criterion for embeddability of manifolds in $\mathbb{R}^m$, Comment. Math. Helv. 72 (1997) 543–555.

[18] A.B. Skopenkov, On the deleted product criterion for embeddability in $\mathbb{R}^m$, Proc. Amer. Math. Soc. 126:8 (1998) 2467–2476.

[19] A.B. Skopenkov, On the Haefliger–Hirsch–Wu invariants for embeddings and immersions, Comment. Math. Helv. 77 (2002) 78–124.

[20] A. Skopenkov, Embedding and Knotting of Higher-Dimensional Manifolds in Euclidean Spaces, London Math. Soc. Lect. Notes, submitted for publication.

[21] J. Segal, A. Skopenkov, S. Spież, Embedding of polyhedra in $\mathbb{R}^m$ and the deleted product obstruction, Topology Appl. 85 (1998) 335–344.

[22] C.T.C. Wall, Theory of Handle Decompositions, Differential Topology, vol. IV, Cambridge University Press, Cambridge, 1964 (mimeographed notes).

[23] C. Weber, Plongements de polyèdres dans le domaine métastable, Comment. Math. Helv. 42 (1967) 1–27.

[24] C. Weber, Deux remarques sur les plongements d'un AR dans un éspace euclidien, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 16 (1968) 851–855.

[25] W.T. Wu, A Theory of Embeddings, Isotopy and Immersions of Polyhedra in Euclidean Space, Science Press, Peking, 1965.

[26] E.C. Zeeman, A proof of the comparison theorem for spectral sequence, Proc. Cambridge Philos. Soc. 53 (1957) 57–62.